Multi-Commodity Flow Based Routing

- Set up ILP formulation for MCF routing
  - Capacity of each edge in $G$ is 2
  - Each edge in $G$ becomes a pair of bi-directional arcs in $F$
  - $n_1 = \{a,l\}$, $n_2 = \{i,c\}$, $n_3 = \{d,f\}$, $n_4 = \{k,d\}$, $n_5 = \{g,h\}$, $n_6 = \{b,k\}$
Flow Network

- Each arc has a cost based on its length
  - Let $x_e^k$ denote a binary variable for arc $e$ w.r.t. net $k$
  - $x_e^k = 1$ means net $k$ uses arc $e$ in its route
  - Total number of $x$-variables: $16 \times 2 \times 6 = 192$

<table>
<thead>
<tr>
<th>arc</th>
<th>cost</th>
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<th>arc</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$(a, b)$</td>
<td>4</td>
<td>$(b, a)$</td>
<td>4</td>
<td>$(b, c)$</td>
<td>8</td>
<td>$(c, b)$</td>
<td>8</td>
</tr>
<tr>
<td>$(d, h)$</td>
<td>4</td>
<td>$(h, d)$</td>
<td>4</td>
<td>$(e, f)$</td>
<td>5</td>
<td>$(f, e)$</td>
<td>5</td>
</tr>
<tr>
<td>$(f, g)$</td>
<td>3</td>
<td>$(g, f)$</td>
<td>3</td>
<td>$(i, j)$</td>
<td>4</td>
<td>$(j, i)$</td>
<td>4</td>
</tr>
<tr>
<td>$(j, k)$</td>
<td>5</td>
<td>$(k, j)$</td>
<td>5</td>
<td>$(k, l)$</td>
<td>3</td>
<td>$(l, k)$</td>
<td>3</td>
</tr>
<tr>
<td>$(a, d)$</td>
<td>7</td>
<td>$(d, a)$</td>
<td>7</td>
<td>$(d, i)$</td>
<td>5</td>
<td>$(i, d)$</td>
<td>5</td>
</tr>
<tr>
<td>$(b, e)$</td>
<td>4</td>
<td>$(e, b)$</td>
<td>4</td>
<td>$(e, h)$</td>
<td>3</td>
<td>$(h, e)$</td>
<td>3</td>
</tr>
<tr>
<td>$(h, j)$</td>
<td>5</td>
<td>$(j, h)$</td>
<td>5</td>
<td>$(f, k)$</td>
<td>8</td>
<td>$(k, f)$</td>
<td>8</td>
</tr>
<tr>
<td>$(c, g)$</td>
<td>4</td>
<td>$(g, c)$</td>
<td>4</td>
<td>$(g, l)$</td>
<td>8</td>
<td>$(l, g)$</td>
<td>8</td>
</tr>
</tbody>
</table>
ILP Objective Function

- Minimize

\[ \begin{align*}
4(x_{a,b}^1 + \cdots + x_{a,b}^6) + 4(x_{b,a}^1 + \cdots + x_{b,a}^6) + 8(x_{b,c}^1 + \cdots + x_{b,c}^6) + \\
8(x_{c,b}^1 + \cdots + x_{c,b}^6) + 4(x_{d,h}^1 + \cdots + x_{d,h}^6) + 4(x_{h,d}^1 + \cdots + x_{h,d}^6) + \\
5(x_{e,f}^1 + \cdots + x_{e,f}^6) + 5(x_{f,e}^1 + \cdots + x_{f,e}^6) + 3(x_{f,g}^1 + \cdots + x_{f,g}^6) + \\
3(x_{g,f}^1 + \cdots + x_{g,f}^6) + 4(x_{i,j}^1 + \cdots + x_{i,j}^6) + 4(x_{j,i}^1 + \cdots + x_{j,i}^6) + \\
5(x_{j,k}^1 + \cdots + x_{j,k}^6) + 5(x_{k,j}^1 + \cdots + x_{k,j}^6) + 3(x_{k,l}^1 + \cdots + x_{k,l}^6) + \\
3(x_{l,k}^1 + \cdots + x_{l,k}^6) + 7(x_{a,d}^1 + \cdots + x_{a,d}^6) + 7(x_{d,a}^1 + \cdots + x_{d,a}^6) + \\
5(x_{d,i}^1 + \cdots + x_{d,i}^6) + 5(x_{i,d}^1 + \cdots + x_{i,d}^6) + 4(x_{b,e}^1 + \cdots + x_{b,e}^6) + \\
4(x_{e,b}^1 + \cdots + x_{e,b}^6) + 3(x_{e,h}^1 + \cdots + x_{e,h}^6) + 3(x_{h,e}^1 + \cdots + x_{h,e}^6) + \\
5(x_{h,j}^1 + \cdots + x_{h,j}^6) + 5(x_{j,h}^1 + \cdots + x_{j,h}^6) + 8(x_{f,k}^1 + \cdots + x_{f,k}^6) + \\
8(x_{k,f}^1 + \cdots + x_{k,f}^6) + 4(x_{c,g}^1 + \cdots + x_{c,g}^6) + 4(x_{g,c}^1 + \cdots + x_{g,c}^6) + \\
8(x_{g,l}^1 + \cdots + x_{g,l}^6) + 8(x_{l,g}^1 + \cdots + x_{l,g}^6) \end{align*} \]
ILP Demand Constraint

- Utilize demand constant
  - $z^k_v = 1$ means node $v$ is the source of net $k$ ($= -1$ if sink)
  - Total number of $z$-constants: $12 \times 6 = 72$

From net $n_1 = \{a, l\}$, we have $z^1_a = 1$, $z^1_l = -1$.
From net $n_2 = \{i, c\}$, we have $z^2_i = 1$, $z^2_c = -1$.
From net $n_3 = \{d, f\}$, we have $z^3_d = 1$, $z^3_f = -1$.
From net $n_4 = \{k, d\}$, we have $z^4_k = 1$, $z^4_d = -1$.
From net $n_5 = \{g, h\}$, we have $z^5_g = 1$, $z^5_h = -1$.
From net $n_6 = \{b, k\}$, we have $z^6_b = 1$, $z^6_k = -1$. 
ILP Demand Constraint (cont)

- Node $a$: source of net $n_1$

\[
\begin{align*}
    x_{a,b}^1 + x_{a,d}^1 - x_{b,a}^1 - x_{d,a}^1 &= 1 \\
    x_{a,b}^2 + x_{a,d}^2 - x_{b,a}^2 - x_{d,a}^2 &= 0 \\
    x_{a,b}^3 + x_{a,d}^3 - x_{b,a}^3 - x_{d,a}^3 &= 0 \\
    x_{a,b}^4 + x_{a,d}^4 - x_{b,a}^4 - x_{d,a}^4 &= 0 \\
    x_{a,b}^5 + x_{a,d}^5 - x_{b,a}^5 - x_{d,a}^5 &= 0 \\
    x_{a,b}^6 + x_{a,d}^6 - x_{b,a}^6 - x_{d,a}^6 &= 0
\end{align*}
\]
ILP Demand Constraint (cont)

- Node $b$: source of net $n_6$

\[
x_{b,a}^1 + x_{b,e}^1 + x_{b,c}^1 - x_{a,b}^1 - x_{e,b}^1 - x_{c,b}^1 = 0
\]
\[
x_{b,a}^2 + x_{b,e}^2 + x_{b,c}^2 - x_{a,b}^2 - x_{e,b}^2 - x_{c,b}^2 = 0
\]
\[
x_{b,a}^3 + x_{b,e}^3 + x_{b,c}^3 - x_{a,b}^3 - x_{e,b}^3 - x_{c,b}^3 = 0
\]
\[
x_{b,a}^4 + x_{b,e}^4 + x_{b,c}^4 - x_{a,b}^4 - x_{e,b}^4 - x_{c,b}^4 = 0
\]
\[
x_{b,a}^5 + x_{b,e}^5 + x_{b,c}^5 - x_{a,b}^5 - x_{e,b}^5 - x_{c,b}^5 = 0
\]
\[
x_{b,a}^6 + x_{b,e}^6 + x_{b,c}^6 - x_{a,b}^6 - x_{e,b}^6 - x_{c,b}^6 = 1
\]
ILP Capacity Constraint

- Each edge in the routing graph allows 2 nets

\[
x_{a,b}^1 + \cdots + x_{a,b}^6 + x_{b,a}^1 + \cdots + x_{b,a}^6 \leq 2
\]

\[
x_{b,c}^1 + \cdots + x_{b,c}^6 + x_{c,b}^1 + \cdots + x_{c,b}^6 \leq 2
\]

\[
x_{d,h}^1 + \cdots + x_{d,h}^6 + x_{h,d}^1 + \cdots + x_{h,d}^6 \leq 2
\]

\[
x_{e,f}^1 + \cdots + x_{e,f}^6 + x_{f,e}^1 + \cdots + x_{f,e}^6 \leq 2
\]

\[
\vdots
\]

\[
x_{h,j}^1 + \cdots + x_{h,j}^6 + x_{j,h}^1 + \cdots + x_{j,h}^6 \leq 2
\]

\[
x_{f,k}^1 + \cdots + x_{f,k}^6 + x_{k,f}^1 + \cdots + x_{k,f}^6 \leq 2
\]

\[
x_{c,g}^1 + \cdots + x_{c,g}^6 + x_{g,c}^1 + \cdots + x_{g,c}^6 \leq 2
\]

\[
x_{g,l}^1 + \cdots + x_{g,l}^6 + x_{l,g}^1 + \cdots + x_{l,g}^6 \leq 2
\]
ILP Solutions

- Min-cost: 108 (= sum of WL), 22 non-zero variable
  - path for $n_1 = \{a, l\}$: we have $x_{a,d}^1, x_{d,h}^1, x_{h,j}^1, x_{j,k}^1, x_{k,l}^1$ assigned 1. Thus, the path is $a \rightarrow d \rightarrow h \rightarrow j \rightarrow k \rightarrow l$. The wirelength is 24.
  - path for $n_2 = \{i, c\}$: we have $x_{i,d}^2, x_{d,a}^2, x_{a,b}^2, x_{b,c}^2$ assigned 1. Thus, the path is $i \rightarrow d \rightarrow a \rightarrow b \rightarrow c$. The wirelength is 24.
  - path for $n_3 = \{d, f\}$: we have $x_{d,h}^3, x_{h,e}^3, x_{e,f}^3$ assigned 1. Thus, the path is $d \rightarrow h \rightarrow e \rightarrow f$. The wirelength is 12.
  - path for $n_4 = \{k, d\}$: we have $x_{k,j}^4, x_{j,i}^4, x_{i,d}^4$ assigned 1. Thus, the path is $k \rightarrow j \rightarrow i \rightarrow d$. The wirelength is 14.
  - path for $n_5 = \{g, h\}$: we have $x_{g,f}^5, x_{f,e}^5, x_{e,h}^5$ assigned 1. Thus, the path is $g \rightarrow f \rightarrow e \rightarrow h$. The wirelength is 11.
  - path for $n_6 = \{b, k\}$: we have $x_{b,c}^6, x_{c,g}^6, x_{g,f}^6, x_{f,k}^6$ assigned 1. Thus, the path is $b \rightarrow c \rightarrow g \rightarrow f \rightarrow k$. The wirelength is 23.
ILP-based MCF Routing Solution

- Net 6 is non-optimal
  - Due to congestion
Drawback of ILP-based Method

- ILP is non-scalable
  - Runtime quickly increases with bigger problem instances
- Shragowitz and Keel presented a heuristic instead
  - Called MM (MiniMax) heuristic [1987]
  - Repeatedly perform shortest path computation and rip-up-and-reroute
MM Heuristic

- Initial set up: shortest path computation
  - Ignore capacity, some paths are not unique

(a) net 1
(b) net 2
(c) net 3
(d) net 4
(e) net 5
(f) net 6
First Iteration of MM Heuristic

- **Step 1**
  - Capacity of channel $c(e,f)$ and $c(d,i)$ is violated
  - Max overflow $M_1 = 3 - 2 = 1 > 0$, so we proceed
  - Notation: channel $c(e,f)$ represents arc pair $(e,f)$ and $(f,e)$

Notation: channel $c(e,f)$ represents arc pair $(e,f)$ and $(f,e)$.
First Iteration of MM Heuristic (cont)

- **Step 2**
  - Set of channels with overflow of \( M_1 \): \( J_1 = \{c(d,i), c(e,f)\} \)
  - Set of channels with overflow of \( M_1 \) and \( M_1 - 1 \): \( J_1^0 = \{c(a,d), c(e,h), c(i,j), c(j,k), c(d,i), c(e,f)\} \)

- **Step 3**
  - Cost of \( J_1^0 = \{c(a,d), c(e,h), c(i,j), c(j,k), c(d,i), c(e,f)\} \) is \( \infty \)
First Iteration of MM Heuristic (cont)

- **Step 4**
  - Set of nets using channels in $J_1$: $K_1 = \{n_1, n_2, n_3, n_4, n_5, n_6\}$
  - Set of nets using channels in $J_1^0$: $K_1^0 = K_1$
First Iteration of MM Heuristic (cont)

- **Step 5**
  - Compute shortest paths for nets in $K_1$ using new cost (= Step 3)
  - $n_1$ & $n_6$ have non-infinity cost, so we proceed
First Iteration of MM Heuristic (cont)

- **Step 6**
  - Net with minimum wirelength increase between \( n_1 \) & \( n_6 \): \( k^0 = n_1 \)

![Diagram](image-url)
First Iteration of MM Heuristic (cont)

- **Step 7**
  - Use new routing for $n_1$
  - Wirelength didn’t change, but congestion improved

![Diagram](image_url)
Second Iteration of MM Heuristic

- Details in the book
  - Use new routing for $n_3$
  - Wirelength increased (due to detour in $n_3$), but congestion improved

![Diagram showing two iterations of MM heuristic](image)